The Mathematical Meaning of Aware's Logo

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Aware's logo has attracted some interest and I have often been asked whether it has a meaning. It does: the logo is a picture of a mathematical theorem in wavelet theory. I will describe the theorem, which is not difficult but does involve many diffferent ideas.

A wavelet basis is defined in terms of a subband (polyphase, multirate) digital filterbank called a *wavelet matrix*. The basis is compactly supported if and only if the filterbank has a finite impulse response. The collection of all wavelet bases is organized by two parameters: the *rank* is the number of bands in the filter. The number of taps in each subband filter is a multiple of the rank, and that multiple is the second parameter – the genus of the filterbank. The collection of all wavelet matrix filterbanks can be organized by rank and genus into a *wavelet matrix spreadsheet*. Each cell in the spreadsheet holds the infinitely many different wavelet matrices (and wavelet bases) that share the same rank and genus.

Aware's logo is a parameter space for real valued wavelet bases of rank 2 and genus 3. Every wavelet matrix in this spreadsheet cell is a 2 x 6 matrix of real numbers:

$$a = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ -a_5 & a_4 & -a_3 & a_2 & -a_1 & a_0 \end{pmatrix} \tag{1}$$

The wavelet basis property is expressed in terms of relationships among the filter taps value, which are required to satisfy 2 linear and 3 quadratic equations:

$$\begin{aligned} a_0 + a_2 + a_4 &= 1 \\ a_1 + a_3 + a_5 &= 1 \\ a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 &= 2 \\ a_0a_2 + a_1a_3 + a_2a_4 + a_3a_5 &= 0 \\ a_0a_4 + a_1a_5 &= 0 \end{aligned}$$

One of these equations is redundant so the tap values a_k can be expressed in terms of two independent parameters. A little calculation rewards us with the formulae

$$
\begin{aligned}
a_0 &= \frac{1}{4}(-2\cos\beta\,\sin\alpha + (1 - \cos\alpha + \sin\alpha)\,(1 + \cos\beta - \sin\beta)) \\
a_1 &= \frac{1}{4}(2\cos\beta\,\sin\alpha + (1 - \cos\alpha - \sin\alpha)\,(1 + \cos\beta \text{ t } \sin\beta)) \\
a_2 &= \frac{1}{2}(1 + \cos(\alpha - \beta) \text{ t } \sin(\alpha - \beta)) \\
a_3 &= \frac{1}{2}(1 + \cos(\alpha - \beta) - \sin(\alpha - \beta)) \\
a_4 &= 1 - \frac{1}{2}(1 + \cos(\alpha - \beta) \text{ t } \sin(\alpha - \mathrm{P})) - \\
&\quad \frac{1}{4}(-2\mathrm{cosp}\,\sin\alpha + (1 - \cos\alpha + \sin\alpha)\,(1 + \cos\beta - \sin\beta)) \\
a_5 &= 1 - \frac{1}{2}(1 + \cos(\alpha - \beta) - \sin(\alpha - \beta)) - \\
&\quad \frac{1}{4}(2\cos\beta\,\sin\alpha + (1 - \cos\alpha - \sin\alpha)\,(1 + \cos\beta \text{ t } \sin\beta))
\end{aligned}
$$

The parameters α and β are angles but they can be thought of as latitude and longitude coordinates of a map: the point (α, β) on the map corresponds to the wavelet basis defined by the wavelet matrix a. Since the tap values have period 2π, we restrict the range of x and β so that $-\pi \leq \alpha < \pi$ and $-\pi \leq \beta < \pi$ and the map is a square. Aware's logo is this map on which each point corresponds to a real wavelet basis of rank 2 and genus 3.

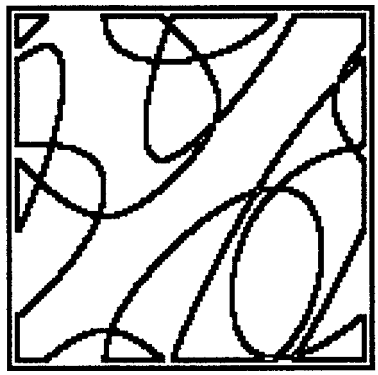
We are not at the the end of our story; indeed, we have only reached the end of the beginning. I must describe how a wavelet basis is built up from a wavelet matrix.

A wavelet basis is an orthonormal collection of real valued functions. Every function ("signal") that has finite energy can be represented as an infinite series in the wavelet basis functions. So far there is nothing new. But wavelet basis functions have two important properties: Every basis function has compact support, and every basis function has a simple expression in terms of one particular basis function called the scaling *funtion.* Properties of the scaling function like continuity and differentiability are inherited by all the other functions in the basis because the simple formula that expresses them in terms of $\varphi(x)$ is continuous and differentiable of all orders.

The scaling function $\varphi(x)$ is defined in terms of its self-similarity by the formula

$$\varphi(x) = \sum_{k=0}^{5} a_k \varphi(2x - k). \tag{2}$$

Now we see how the wavelet matrix and the filter tap values a_k come into the picture. Apart from a few peculiar cases (the one that give joy to mathematicians), the scaling function is uniquely determined by the wavelet matrix. The scaling function is compactly supported. In fact, for rank 2 and genus 5 the scaling function is zero outside the interval $[0, 5]$.



Now we have reached the beginning of the end of our story. Once we have the scaling function we study its properties. One property that is important for applications is differentiabilty. Is $\varphi(x)$ differentiable? If it were, then we could differentiate the scaling equation eq(2) and we would find

$$\frac{d\varphi}{dx}(x) = \sum_{k=0}^{5} 2a_k \frac{d\varphi}{dx}(2x - k), \tag{3}$$

and if we differentiated the scaling equation s times we would find

$$\frac{d^s\varphi}{dx^s}(x) = \sum_{k=0}^{5} 2^s a_k \frac{d^s\varphi}{dx^s}(2x - k). \tag{4}$$

This equation is like the original scaling equation itself except that each filter tap value has been multiplied by 2^s; can we still solve this equation?

If $\varphi(x)$ is s-fold differentiable, then we certainly can solve this equation and find the derivative. It turns out that the equation can also be solved for other special values of s that depend on the wavelet matrix a and therefore on the wavelet matrix parameters x, y. For these solutions $\varphi(x)$ is not actually differentiable but it behaves in many ways as if it were. Remarkably enough, these other values are not always integers. *So we* have discovered a concept of formal *differentiability* that includes the usual derivatives but also makes place for "fractional" differentiation.

If we ask which values of the parameters lead to wavelet bases that have an ordinary *or* a formal first derivative, then we find that these bases lie on certain curves in the parameter space. These are the curves in the logo that separate the black and the white regions. *The curves on the logo identify all real wavelet basis functions of rank 2* and *genus 3 that are formally differentiable.*

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Figure 1: Aware's logo. The square is a map of the two dimensional pinched torus parameter space of six coefficient wavelet matrices. The curves *are* the loci of wavelet matrices that correspond to wavelet bases that are formally differentiable of order 1.